Linear and non-linear equations

Operator notation –

An *operator* is a mathematical operation (or set of operations) which acts on a function.

For example, the operator $D = \frac{d}{dx}$ is an operator which differentiates a function $f(x)$.

i.e. $D(f(x)) = \frac{df}{dx}$

The operator $M = 2*$ might be the operator which multiplies by 2 –

$M(f(x)) = 2f(x)$.

Def: An operator, O, is *linear* if $O(c_1 f_1(x) + c_2 f_2(x)) = c_1 O(f_1(x)) + c_2 O(f_2(x))$. Otherwise it is *non-linear*.

* where c_1, c_2 = constants

Ex – Differentiation of a function of one variable is linear –

$$\frac{d}{dx}\{c_1 f_1 + c_2 f_2\} = c_1 \frac{df_1}{dx} + c_2 \frac{df_2}{dx}$$

integration is linear –

$$\int \{c_1 f_1(x) + c_2 f_2(x)\} dx = \int c_1 f_1(x) dx + \int c_2 f_2(x) dx$$
$$= c_1 \int f_1(x) dx + c_2 \int f_2(x) dx$$

But the operator $Q(y) = y\frac{dy}{dx}$ acting on a function $y = y(x)$ is non-linear because

$$Q(c_1 y_1 + c_2 y_2) = (c_1 y_1 + c_2 y_2)\frac{d}{dx}(c_1 y_1 + c_2 y_2)$$
$$= (c_1 y_1 + c_2 y_2)\left\{c_1 \frac{dy_1}{dx} + c_2 \frac{dy_2}{dx}\right\}$$
$$= c_1^2 y_1 \frac{dy_1}{dx} + c_2^2 y_2 \frac{dy_2}{dx} + c_1 c_2 y_1 \frac{dy_2}{dx} + c_1 c_2 y_2 \frac{dy_1}{dx}$$
$$\neq c_1 y_1 \frac{dy_1}{dx} + c_2 y_2 \frac{dy_2}{dx}$$

A differential equation can be classified as linear or non-linear by thinking of the equation as an operator. If you replace the unknown function $y = y(x)$ in the equation by the linear combination $c_1 y_1 + c_2 y_2$ you can determine linearity —

Def:

An ODE is linear and homogeneous if it is of the form

$L(y) = 0$ where L is a linear operator.

An ODE is linear and non-homogeneous if it is of the form

$L(y) = f(x)$ where L is a linear operator.

It can be shown (and is fairly easy to see) that any linear ODE must be of the form –

$$a_n(x)\,y^{(n)}(x) + a_{n-1}(x)\,y^{(n-1)}(x) + \dots + a_2(x)\,y''(x) + a_1(x)\,y'(x) + a_0(x)\,y(x) = f(x)$$

where $a_0(x), a_1(x), \dots, a_n(x)$ are all functions of x alone.

If $f(x) = 0$ the equation is homogeneous. If $f(x) \neq 0$ the equation is non-homogeneous.

Most of the equations that we can solve (and most of what you see as an undergrad) are linear.

A nice property of linear homogeneous ODEs is that solutions are additive. That is, if y_1 and y_2 are both solutions of the equation $L(y) = 0$, that is $L(y_1) = 0$ and $L(y_2) = 0$, then $y = c_1 y_1 + c_2 y_2$ is also a solution of the equation, that is $L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2) = 0$ because L is linear.

Ex: Consider $\ddot{x} = -x$ or $\ddot{x} + x = 0$ or $\frac{d^2x}{dt^2} + x = 0$

This ODE is linear homogeneous.

You can show that $x = \cos t$ is a solution and $x = \sin t$ is a solution. Therefore, since the equation is linear then $x = c_1 \cos t + c_2 \sin t$ is also a solution. In fact, it is the general solution!

The other important property of linear equations we will use is that if we can find a general solution, y_g, of the linear equation $L(y) = 0$ and a particular solution, y_p, of the non-linear equation $L(y) = f(x)$ then the solutions are additive and $y = y_g + y_p$ is the general solution of the non-homogeneous equation.

that is, if $L(y_g)=0$ and $L(y_p)=f(x)$, then

$$L(y_g+y_p) = L(y_g)+L(y_p) \quad \text{(why?)}$$
$$= 0 + f(x)$$
$$= f(x)$$

So we will reduce the problem of solution of a non-homogeneous linear ODE to two hopefully simpler equations. This will be a recurring theme throughout this course.

Ex! Consider $y''+4y=12x$ which is a linear non-homogeneous ODE.

First we consider the homogeneous equation –

$$y''+4y=0$$

You can show that $y_1=\cos 2x$ and $y_2=\sin 2x$ are both solutions of $y''+4y=0$. Since this equation is linear, then

$$y_g = c_1 y_1 + c_2 y_2$$
$$= c_1\cos 2x + c_2 \sin 2x$$

is the general solution to the homogeneous equation.

You can then show that $y_p = 3x$ is a solution of the non-homogeneous equation.

Therefore $y = y_g + y_p$

$$= c_1 \cos 2x + c_2 \sin 2x + 3x$$

is the general solution of the non-homogeneous equation $y''+4y=12x$

Existence and uniqueness –

These 2 questions are fundamental. The first question, that is "does a solution exist?", is not important in applications, if we can find a solution then it exists.

The second question, "is this solution unique?" is important but you can imagine that if a physical problem is well posed with sufficient initial or boundary conditions then a solution must be unique. Otherwise the world be ~~very~~ a very strange place.

Uniqueness is, in general, hard to prove. Most books give one general theorem and say "the proof is beyond the level of this course". I prefer we address the cases we can as we proceed.

So, as a first example, consider the initial value problem

$$y' = f(x,y) \ , \ y(x_0) = y_0$$

First notice some things —

i) If y exists for some interval $a < x < b$ then y is a differentiable function of x on the interval (a,b).

ii) If i) is true then $f(x,y)$ is itself a function of x. It may be implicit but it is a function of x, i.e. if f is a function of x and y and y is a function of x then f is a function of x.

So, in fact, the 1st order initial value problem is equivalent to the problem

$$y' = F(x) \ , \ y(x_0) = y_0$$

So now for uniqueness —

Suppose $y_1 = y_1(x)$ and $y_2 = y_2(x)$ are both solutions on some (a,b) and $x_0 \in (a,b)$

that is $y_1' = F(x)$, $y_1(x_0) = y_0$ and $y_2' = F(x)$, $y_2(x_0) = y_0$

Well then

$$y_1 = \int F(x)dx + C_1$$

and $y_2 = \int F(x)dx + C_2$

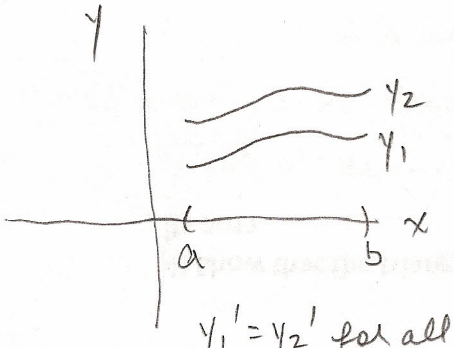
or $y_1 = y_2 + C$, $C =$ any constant

but if $y_1(x_0) = y_0$

and $y_2(x_0) = y_1(x_0) - C = y_0$

then $C = 0$ and $y_1(x) = y_2(x)$ and the solution is unique on (a,b)

Intuitively this just says that if two functions, $y_1(x)$ and $y_2(x)$ have the same derivative everywhere on some interval (a,b) then they look the same and can, at most, be shifted vertically —



$y_1' = y_2'$ for all $x \in (a,b)$

But if they are equal at any x in (a,b) they have to be the same —